Lafayette Problem Group
Problem Set 5

Try as many of these as you can by the next meeting, which will be ***Thursday, October 10, from 4 to 5pm in Pardee 218 (Math Common Room)****. Good Luck!.*

Problem 1: In a round-robin, every team plays every other team once. In any round-robin tournament, the total number of wins by all teams equals the total number of losses – do you see why? Once you do, determine the total number of wins if n teams are involved.

Problem 2: As a follow-up to Problem 1, can you show that for any round-robin tournament, the sum of the squares of the number of wins by each team is equal to the the sum of the squares of the number of losses by each team?

For example, suppose there are four teams, with team A beating B, C, and D, and with B beating C, C beating D, and D beating B. Then the wins would give $3^2 + 1^2 + 1^2 + 1^2 = 12$, and the losses would give $0^2 + 2^2 + 2^2 + 2^2 = 12$.

Problem 3: The Birthday Problem is a classical problem in probability. Suppose there are 25 people in a room. Ignoring leap years, what is the probability that at least two people in the room share a birthday? Can you generalize this to n people?

Now, get the 25 people to stand in a line. What is the probability at least two people standing next to each other share a birthday? Can you generalize this to n people? Then, get the 25 people to stand on a 5×5 grid. What is the probability at least two people standing next to each other (front to back, left to right, or on diagonals) share a birthday? Can you generalize this to an $n \times m$ grid of people?

Problem 4: Let

$$A = \{x \mid |2x - x^2| \leq x\},$$

$$B = \{x \mid |\frac{x}{1-x}| \leq \frac{x}{1-x}\}, \text{ and}$$

$$C = \{x \mid ax^2 + x + b < 0\}.$$

If $(A \cup B) \cap C = \emptyset$ and $(A \cup B) \cup C = \mathbb{R}$, find a and b.

Turn over!

Problem 5: Given a linear fractional transformation of x into $f_1(x) = \frac{2x-1}{x+1}$, define $f_{n+1}(x) = f_1(f_n(x))$ for all positive integers n. It can be shown that $f_{35} = f_5$. Determine A, B, C, and D so that $f_{28}(x) = \frac{Ax+B}{Cx+D}$.

Problem 6[1]**:** Given a positive integer n, what is the largest k such that the numbers $1, 2, ..., n$ can be put into k boxes so that the sum of the numbers in each box is the same? (When $n = 8$, the example $\{1, 2, 3, 6\}, \{4, 8\}, \{5, 7\}$ shows that the largest k is at least 3.)

Problem 7[2]**:** Let S be the class of functions from $[0, \infty)$ to $[0, \infty)$ that satisfies:

1. The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x + 1)$ are in S.
2. If $f(x)$ and $g(x)$ are in S, the functions $f(x) + g(x)$ and $f(g(x))$ are in S.
3. If $f(x)$ and $g(x)$ are in S and $f(x) \geq g(x)$ for all $x \geq 0$, then the function $f(x) - g(x)$ is in S.

Prove that if $f(x)$ and $g(x)$ are in S, then the function $f(x)g(x)$ is also in S.

And the problem we are still working on . . . Any four points p_1, p_2, p_3, p_4 in the plane create six distances between pairs of points. Can you find four points in the plane so that all six distances are different integers and so that no three points lie on the same line?

The answer is yes! It's possible to do it with a trapezoid whose parallel sides are distance 40 away from each other. Can you construct this example?

But, as is very common in mathematics, this just leads to further questions:

1. Are there other examples? Infinitely many other examples?
2. What if no two sides can be parallel?
3. What if you ask the same questions about five points and the ten distances between pairs of them?
4. *Insert your further question here!*

[1] This problem is based on a former Putnam exam, a wickedly-difficult national math competition on the first Saturday of every December. This year's exam will take place from 10am until 6pm on **Saturday, December 7**. Don't worry if the problem is too hard - the median score on the Putnam exam is usually something like 1 out of 120. So you're never expected to do too well, which is a liberating concept. Just do it for fun!

[2] Another Putnam problem!