Lafayette Problem Group
Problem Set 4

Try as many of these as you can by the next meeting, which will be ***Thursday, October 3, from 4 to 5pm in Pardee 218 (Math Common Room)****. Good Luck!.*

Problem 1: Prove that the product of any four consecutive integers is one less than a perfect square.

Problem 2: Show that for all positive integers n, 14 divides $3^{4n+2} + 5^{2n+1}$.

Problem 3: For $x > 0$, show that $e^x < (1+x)^{1+x}$.

Problem 4: If you have an unlimited supply of m cent stamps and n cent stamps, where m and n are relative prime (so they don't have a common factor greater than 1), what is the largest value of postage that you can't make with the stamps? (Last week, we saw that when $m = 7$ and $n = 10$, the largest such value is 53, which just happens to be $(6 \times 9) - 1$.)

Problem 5: Let

$$A = \{x \mid |2x - x^2| \leq x\},$$

$$B = \{x \mid |\frac{x}{1-x}| \leq \frac{x}{1-x}\}, \text{ and}$$

$$C = \{x \mid ax^2 + x + b < 0\}.$$

If $(A \cup B) \cap C = \emptyset$ and $(A \cup B) \cup C = \mathbb{R}$, find a and b.

Problem 6: Given a linear fractional transformation of x into $f_1(x) = \frac{2x-1}{x+1}$, define $f_{n+1}(x) = f_1(f_n(x))$ for all positive integers n. It can be shown that $f_{35} = f_5$. Determine A, B, C, and D so that $f_{28}(x) = \frac{Ax+B}{Cx+D}$.

Turn over!

Problem 7[1]**:** Given a positive integer n, what is the largest k such that the numbers $1, 2, ..., n$ can be put into k boxes so that the sum of the numbers in each box is the same? (When $n = 8$, the example $\{1, 2, 3, 6\}, \{4, 8\}, \{5, 7\}$ shows that the largest k is at least 3.)

Problem 8[2]**:** Let S be the class of functions from $[0, \infty)$ to $[0, \infty)$ that satisfies:

1. The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x + 1)$ are in S.
2. If $f(x)$ and $g(x)$ are in S, the functions $f(x) + g(x)$ and $f(g(x))$ are in S.
3. If $f(x)$ and $g(x)$ are in S and $f(x) \geq g(x)$ for all $x \geq 0$, then the function $f(x) - g(x)$ is in S.

Prove that if $f(x)$ and $g(x)$ are in S, then the function $f(x)g(x)$ is also in S.

And the problem we are still working on ... Any four points p_1, p_2, p_3, p_4 in the plane create six distances between pairs of points. Can you find four points in the plane so that all six distances are different integers and so that no three points lie on the same line?

The answer is yes! It's possible to do it with a trapezoid whose parallel sides are distance 40 away from each other. Can you construct this example?

But, as is very common in mathematics, this just leads to further questions:

1. Are there other examples? Infinitely many other examples?
2. What if no two sides can be parallel?
3. What if you ask the same questions about five points and the ten distances between pairs of them?
4. *Insert your further question here!*

[1] This problem is based on a former Putnam exam, a wickedly-difficult national math competition on the first Saturday of every December. This year's exam will take place from 10am until 6pm on **Saturday, December 7**. Don't worry if the problem is too hard - the median score on the Putnam exam is usually something like 1 out of 120. So you're never expected to do too well, which is a liberating concept. Just do it for fun!

[2] Another Putnam problem!